Finite Larmor Radius and Compressibility Effects on Thermosolutal Instability of a Plasma

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The thermosolutal instability of a compressible plasma due to the effects of the ion Larmor radius is considered. The system is found to be stable for $(C_p/g)\,\beta < 1$. The finite Larmor radius and the compressibility introduce oscillatory modes in the system for $(C_p/g)\,\beta > 1$. The compressibility, solute gradient and finite Larmor radius stabilize the stationary convection. The necessary conditions for overstability are also derived.

1. Introduction

The thermal instability of a fluid layer, heated from below, has been discussed by Chandrasekhar [1] under various assumptions. The stabilizing influence of a finite Larmor radius, resulting in a magnetic viscosity, on plasma instabilities has been demonstrated by many authors [2–5]. Melchior and Popowich [6] have considered the finite Larmor radius effect on the Kelvin-Helmholtz instability of a fully ionized plasma, while the effect on the Rayleigh-Taylor instability has been studied by Singh and Hans [7]. Sharma [8] has studied the effect of a finite Larmor radius on the thermal instability of a plasma. The problem of thermohaline convection in a layer of fluid heated from below and subjected to a stable salinity gradient has been investigated by Veronis [9]. For thermal and thermohaline convection problems, the Boussinesq approximation has been used, which is well justified in the case of incompressible fluids.

When the fluids are compressible, the equations governing the system become quite complicated. Spiegel and Veronis [10] have simplified the set of equations governing the flow of compressible fluids under the following assumptions:

(a) the depth of the fluid layer is much smaller than the scale height as defined by them, if only motions of infinitesimal amplitudes are considered, and

(b) the fluctuations in density, temperature and pressure, introduced by the motion, do not exceed their static variations, in non-linear investigations.

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Under the above assumption, Spiegel and Veronis [10] have found that the equations governing convection in a perfect gas are the same as those for incompressible fluids except that the static temperature gradient is replaced by its excess over the adiabatic one, and C_v is replaced by C_p. Using these approximations, Sharma [11] has studied the thermal instability in compressible fluids in the presence of rotation and a magnetic field. In another study, Sharma and Sharma [12] have considered the thermosolutal instability of a plasma including the finite Larmor radius effect.

In the stellar case, the physics is quite similar to Veronis [9] thermohaline configuration, in that helium acts like salt in raising the density and in diffusing more slowly than heat. The problem of the onset of thermal instability in the presence of a solute gradient is of great importance because of its application to atmospheric physics and astrophysics, especially in the case of the ionosphere and the outer layer of the solar atmosphere (Spiegel [13]). The finite Larmor radius and compressibility effects are likely to be important in these regions. The present paper therefore considers the thermosolutal instability of a plasma including compressibility and finite Larmor radius effects.

2. Perturbation Equations and Dispersion Relation

Consider an infinite, horizontal, compressible, viscous, and conducting plasma layer of depth d, heated and soluted from below so that the temperatures, density and solute concentrations at the bottom surface $z = 0$ are T_0, ϱ_0, and C_0 and at the upper surface $z = d$ are T_d, ϱ_d, and C_d, respectively,

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and that a uniform temperature gradient $\beta\ (=|\mathrm{d}T/\mathrm{d}z|)$ and uniform solute gradient $\beta'\ (=|\mathrm{d}C/\mathrm{d}z|)$ are maintained. The gravity force $\boldsymbol{g}(0,0,-g)$ and uniform field $\boldsymbol{H}(0,0,H)$ pervade the system.

Spiegel and Veronis [10] defined f as any of the state variables (pressure (p), density (ϱ) or temperature (T)) and expressed these in the form

$$f(x,y,z,t)=f_m+f_0(z)+f'(x,y,z,t)\,, \tag{1}$$

where f_m is the constant space average of f, f_0 is the variation in the absence of motion and f' is the fluctuation resulting from motion.

The initial state is therefore a state in which the density, pressure, temperature, solute concentration and velocity at any point in the plasma are given by

$$\varrho=\varrho(z),\quad p=p(z),\quad T=T(z),\quad C=C(z),\quad \boldsymbol{v}=0, \tag{2}$$

respectively, where

$$\begin{aligned}
&T(z)=T_0-\beta z,\quad C(z)=C_0-\beta' z,\\
&p(z)=p_m-g\int_0^z(\varrho_m+\varrho_0)\,\mathrm{d}z,\\
&\varrho(z)=\varrho_m[1-\alpha_m(T-T_m)\\
&\qquad\quad+\alpha'_m(C-C_m)+K_m(p-p_m)],
\end{aligned}\tag{3}$$

$$\alpha_m=-\left(\frac{1}{\varrho}\frac{\partial\varrho}{\partial T}\right)_m,\quad \alpha'_m=-\left(\frac{1}{\varrho}\frac{\partial\varrho}{\partial C}\right)_m,$$

$$K_m=\left(\frac{1}{\varrho}\frac{\partial\varrho}{\partial p}\right)_m.$$

The linearized hydromagnetic perturbation equations appropriate to the problem are

$$\frac{\partial\boldsymbol{v}}{\partial t}=-\left(\frac{1}{\varrho_m}\right)\nabla\delta p-\left(\frac{1}{\varrho_m}\right)\nabla\mathbf{P}+\nu\nabla^2\boldsymbol{v}+\frac{\mu_e}{4\pi\varrho_m}(\nabla\times\boldsymbol{h})\times\boldsymbol{H}+\boldsymbol{g}\left(\frac{\delta\varrho}{\varrho_m}\right), \tag{4}$$

$$\nabla\cdot\boldsymbol{v}=0, \tag{5}$$

$$\frac{\partial\theta}{\partial t}=\left(\beta-\frac{g}{C_p}\right)\omega+\varkappa\nabla^2\theta, \tag{6}$$

$$\frac{\partial\gamma}{\partial t}=\beta'\omega+\varkappa'\nabla^2\gamma, \tag{7}$$

$$\frac{\partial\boldsymbol{h}}{\partial t}=(\boldsymbol{H}\cdot\nabla)\,\boldsymbol{v}+\eta\nabla^2\boldsymbol{h}, \tag{8}$$

$$\nabla\cdot\boldsymbol{h}=0, \tag{9}$$

where $\delta\varrho$, δp, $\boldsymbol{v}(u,v,w)$, $\boldsymbol{h}(h_x,h_y,h_z)$, θ and γ denote, respectively, the perturbations in ϱ and p, the velocity, and the perturbations in the magnetic field $\boldsymbol{H}$, T and C. μ, $\nu\ (=\mu/\varrho_m)$, μ_e, $\varkappa$, $\varkappa'$, g/C_p, η and $\mathbf{P}$ stand for viscosity, kinematic viscosity, magnetic permeability, thermal diffusivity, solute diffusivity, adiabatic gradient, resistivity and stress tensor taking into account the finite Larmor radius effects, respectively.

The equation of state

$$\varrho=\varrho_m[1-\alpha(T-T_m)+\alpha'(C-C_m)] \tag{10}$$

contains the thermal coefficient of expansion α and an analogous solute coefficient α'. The change in density is caused mainly by the temperature and solute concentration, and the suffix m refers to values at the reference level $z=0$. The change in density $\delta\varrho$, caused by the perturbations θ and γ is given by

$$\delta\varrho=-\varrho_m(\alpha\theta-\alpha'\gamma)\,. \tag{11}$$

For the magnetic field along the z-axis the stress tensor $\mathbf{P}$, taking into account the finite ion gyration radius (Vandakurov [4]), has the components

$$\begin{aligned}
P_{xx}&=-\varrho_m\nu_0\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right),\\
P_{xy}&=P_{yx}=\varrho_m\nu_0\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right),\\
P_{xz}&=P_{zx}=-2\varrho_m\nu_0\left(\frac{\partial v}{\partial z}+\frac{\partial \omega}{\partial y}\right),\\
P_{yy}&=\varrho_m\nu_0\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right),\\
P_{yz}&=P_{zy}=2\varrho_m\nu_0\left(\frac{\partial \omega}{\partial x}+\frac{\partial u}{\partial z}\right),\quad P_{zz}=0,
\end{aligned}\tag{12}$$

where $\varrho_m\nu_0=NT/4\omega_{\mathrm{H}}$, ω_{H} being the ion gyration frequency, while N and T are the number density and temperature of the ions, respectively.

Analyzing the disturbances into normal modes, we assume that the perturbation quantities are of the form

$$\begin{aligned}
&[\omega,h_z,\zeta,\xi,\theta,\gamma]\\
&\quad=[W(z),K(z),Z(z),X(z),\Theta(z),\Gamma(z)]\\
&\qquad\cdot\exp(i k_x x+i k_y y+n t)\,,
\end{aligned}\tag{13}$$

where k_x and k_y are the wave numbers in the x and y directions, respectively, $k = (k_x^2 + k_y^2)^{1/2}$ is the resultant wave number, and n is the growth rate, which is, in general, a complex constant.

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \quad \text{and} \quad \xi = \frac{\partial h_y}{\partial x} - \frac{\partial h_x}{\partial y}$$

stand for the z-components of vorticity and current density, respectively.

Expressing the coordinates x, y, z in the new unit of length d and letting $a = kd$, $\sigma = nd^2/\nu$, $p_1 = \nu/\varkappa$, $p_2 = \nu/\eta$, $q = \nu/\varkappa'$, $G = C_p \beta/g$ and $D = \mathrm{d}/\mathrm{d}z$, (4)–(9), with the help of (11)–(13) in nondimensional form, become

$$(D^2 - a^2)(D^2 - a^2 - \sigma)\, W - \left(\frac{g d^2}{\nu}\right) a^2 (\alpha \Theta - \alpha' \Gamma)$$

$$+ \frac{\mu_e H d}{4\pi \varrho_m \nu} (D^2 - a^2)\, DK - \left(\frac{v_0 d}{\nu}\right) (2D^2 + a^2)\, DZ = 0, \tag{14}$$

$$(D^2 - a^2 - \sigma)\, Z = - \frac{\mu_e H d}{4\pi \varrho_m \nu} DX - \left(\frac{v_0}{\nu d}\right) (2D^2 + a^2)\, DW, \tag{15}$$

$$(D^2 - a^2 - p_1 \sigma)\, \Theta = - \frac{d^2}{\varkappa} \left(\beta - \frac{g}{C_p}\right) W, \tag{16}$$

$$(D^2 - a^2 - q\sigma)\, \Gamma = - \frac{\beta' d^2}{\varkappa'} W, \tag{17}$$

$$(D^2 - a^2 - p_2 \sigma)\, K = - \left(\frac{H d}{\eta}\right) DW, \tag{18}$$

$$(D^2 - a^2 - p_2 \sigma)\, X = - \left(\frac{H d}{\eta}\right) DZ. \tag{19}$$

Consider the case where both boundaries are free as well as perfect conductors of both heat and solute, while the adjoining medium is electrically nonconducting. The boundary conditions for this case, using (13), are

$$\left.\begin{array}{l} W = D^2 W = DZ = X = \Theta = \Gamma = 0 \\ \text{and } \boldsymbol{h} \text{ are continuous} \end{array}\right\} \text{ at } z = 0 \text{ and } 1. \tag{20}$$

Eliminating Z, K, X, Θ and Γ between (14)–(19) and substituting the proper solution $W = W_0 \sin \pi z$, W_0 being a constant, in the resultant equation, we obtain the dispersion relation

$$R_1 x = \left(\frac{G}{G-1}\right) \Bigg[(1+x)(1+x+i\sigma_1)(1+x+ip_1\sigma_1) + S_1 x \frac{(1+x+ip_1\sigma_1)}{(1+x+iq\sigma_1)} + \frac{Q_1 (1+x)(1+x+ip_1\sigma_1)}{(1+x+ip_2\sigma_1)} + \frac{U(2-x)^2 (1+x+ip_1\sigma_1)(1+x+ip_2\sigma_1)}{(1+x+i\sigma_1)(1+x+ip_2\sigma_1) + Q_1} \Bigg], \tag{21}$$

where

$$R_1 = \frac{g \alpha \beta d^4}{\nu \varkappa \pi^4}, \quad S_1 = \frac{g \alpha' \beta' d^4}{\nu \varkappa' \pi^4},$$

$$Q_1 = \frac{\mu_e H^2 d^2}{4\pi \varrho_m \nu \eta \pi^2} \quad \text{and} \quad U = \frac{v_0^2}{\nu^2}.$$

3. Stability of the System and Oscillatory Modes

Multiplying (14) by W^*, the complex conjugate of W, and using (15)–(19) together with the boundary conditions (20), we obtain

$$(I_1 + \sigma I_2) + \frac{g \alpha' \varkappa' a^2}{\nu \beta'} (I_5 + q \sigma^* I_6) + d^2 (I_7 + \sigma^* I_8) + \frac{\mu_e \eta d^2}{4\pi \varrho_m \nu} (I_9 + p_2 \sigma I_{10}) + \frac{\mu_e \eta}{4\pi \varrho_m \nu} (I_{11} + p_2 \sigma^* I_{12}) = \frac{C_p \alpha \varkappa a^2}{\nu (G-1)} (I_3 + p_1 \sigma^* I_4), \tag{22}$$

where

$$I_1 = \int_0^1 (|D^2 W|^2 + 2a^2 |DW|^2 + a^4 |W|^2)\, \mathrm{d}z,$$

$$I_2 = \int_0^1 (|DW|^2 + a^2 |W|^2)\, \mathrm{d}z,$$

$$I_3 = \int_0^1 (|D\Theta|^2 + a^2 |\Theta|^2)\, \mathrm{d}z, \quad I_4 = \int_0^1 |\Theta|^2\, \mathrm{d}z,$$

$$I_5 = \int_0^1 (|D\Gamma|^2 + a^2 |\Gamma|^2)\, \mathrm{d}z, \quad I_6 = \int_0^1 |\Gamma|^2\, \mathrm{d}z,$$

$$I_7 = \int_0^1 (|DZ|^2 + a^2 |Z|^2)\, \mathrm{d}z, \quad I_8 = \int_0^1 |Z|^2\, \mathrm{d}z,$$

$$I_9 = \int_0^1 (|DX|^2 + a^2 |X|^2)\, \mathrm{d}z, \quad I_{10} = \int_0^1 |X|^2\, \mathrm{d}z,$$

$$I_{11} = \int_0^1 (|D^2K|^2 + 2a^2|DK|^2 + a^4|K|^2)\,\mathrm{d}z,$$

$$I_{12} = \int_0^1 (|DK|^2 + a^2|K|^2)\,\mathrm{d}z. \tag{23}$$

The integrals $I_1 - I_{12}$ are all positive definite. Putting $\sigma = \sigma_r + i\sigma_i$ and equating the real and imaginary parts of (22), we obtain

$$\left[I_2 + \frac{g\,\alpha'\,\varkappa'\,a^2}{\nu\,\beta'}\,q\,I_6 + \frac{\mu_e\,\eta}{4\pi\,\varrho_m\,\nu}\,p_2\,I_{12} + \frac{\mu_e\,\eta\,d^2}{4\pi\,\varrho_m\,\nu}\,p_2\,I_{10} + d^2 I_8 + \frac{C_p\,\alpha\,\varkappa\,a^2}{\nu(1-G)}\,p_1\,I_4\right]\sigma_r = -\left[I_1 + \frac{g\,\alpha'\,\varkappa'\,a^2}{\nu\,\beta'}\,I_5 + \frac{\mu_e\,\eta}{4\pi\,\varrho_m\,\nu}\,I_{11} + \frac{\mu_e\,\eta\,d^2}{4\pi\,\varrho_m\,\nu}\,I_9 + d^2 I_7 + \frac{C_p\,\alpha\,\varkappa\,a^2}{\nu(1-G)}\,I_3\right], \tag{24}$$

and

$$\left[I_2 - \frac{g\,\alpha'\,\varkappa'\,a^2}{\nu\,\beta'}\,q\,I_6 - \frac{\mu_e\,\eta}{4\pi\,\varrho_m\,\nu}\,p_2\,I_{12} + \frac{\mu_e\,\eta\,d^2}{4\pi\,\varrho_m\,\nu}\,p_2\,I_{10} - d^2 I_8 + \frac{C_p\,\alpha\,\varkappa\,a^2}{\nu(G-1)}\,p_1\,I_4\right]\sigma_i = 0. \tag{25}$$

It is evident from (24) that σ_r is negative if $G < 1$. The system is therefore stable for $G < 1$. It is clear from (25) that, for $G > 1$, σ_i may be zero or nonzero, meaning that the modes may be nonoscillatory or oscillatory. The oscillatory modes are introduced due to the presence of a magnetic field, a finite Larmor radius and a solute gradient. In the absence of a magnetic field and solute gradient, the oscillatory modes are not allowed for $G > 1$, but in the presence of solute gradient, magnetic field and finite Larmor radius effects, the oscillatory modes come into play.

4. The Case of Stationary Convection

For the stationary convection one has $\sigma = 0$ and (21) reduces to

$$R_1 = \left(\frac{G}{G-1}\right)\left[\left(\frac{1+x}{x}\right)\{(1+x)^2 + Q_1\} + \frac{U(2-x)^2(1+x)^2}{x\{(1+x)^2+Q_1\}} + S_1\right], \tag{26}$$

which expresses the modified Rayleigh number R_1 as a function of the dimensionless wave number x and the parameters S_1, Q_1, U and G. For fixed values of Q_1, S_1 and U, let the nondimensional number G accounting for the compressibility effects be also kept as fixed. Then we find that

$$\bar{R}_c = \left(\frac{G}{G-1}\right) R_c, \tag{27}$$

where R_c and $\bar{R}_c$ denote the critical Rayleigh numbers in the absence and presence of compressibility. The effect of compressibility is, thus, to postpone the onset of thermal instability. The case $G > 1$ is relevant here as $G = 1$ and $G < 1$ correspond to infinite and negative Rayleigh numbers. Hence we obtain a stabilizing effect of compressibility. It is evident from (26) that

$$\frac{\mathrm{d}R_1}{\mathrm{d}U} = \left(\frac{G}{G-1}\right)\frac{(2-x)^2(1+x)^2}{x\{(1+x)^2+Q_1\}}, \tag{28}$$

and

$$\frac{\mathrm{d}R_1}{\mathrm{d}S_1} = \left(\frac{G}{G-1}\right), \tag{29}$$

which are positive. The stable solute gradient and finite Larmor radius, therefore, stabilize thermosolutal instability of a plasma.

5. The Case of Overstability

Here we discuss the possibility as to whether instability may occur as overstability. Since for overstability our aim is to determine the critical Rayleigh number for the onset of instability via a state of pure oscillations, it will suffice to find conditions for which (21) will admit of solutions with σ_1 real. Equating real and imaginary parts of (21) and eliminating R_1 between them, we obtain

$$A_4 c^4 + A_3 c^3 + A_2 c^2 + A_1 c + A_0 = 0, \tag{30}$$

where we have written $c = \sigma_1^2$, $b = 1 + x$ and

$$A_4 = p_2^4 q^2 (1 + p_1)\, b, \tag{31}$$

$$A_0 = b^2 (b^2 + Q_1)^2 [(1 + p_1)\, b^3 + S_1 (p_1 - q)(b - 1) + Q_1 (p_1 - p_2)\, b] + U b^4 (3 - b)^2 [(p_1 - 1)\, b^2 + Q_1 (p_1 + p_2)]. \tag{32}$$

The four values of c, σ_1 being real, are positive. The product of the roots is A_0/A_4, which is positive if $A_0 > 0$ (since from (31), $A_4 > 0$). It is clear from (32) that A_0 is always positive if

$$p_1 > p_2, \quad p_1 > q, \quad p_1 > 1. \tag{33}$$

This means that

$$\varkappa < \eta, \quad \varkappa < \varkappa' \quad \text{and} \quad \varkappa < \nu. \tag{34}$$

$\varkappa < \eta$, $\varkappa < \varkappa'$ and $\varkappa < \nu$ are thus the necessary conditions for the existence of overstability.

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